



Alexandria University  
**Alexandria Engineering Journal**

[www.elsevier.com/locate/aej](http://www.elsevier.com/locate/aej)  
[www.sciencedirect.com](http://www.sciencedirect.com)



# On comparing and clustering the spectral densities of several almost cyclostationary processes



Mohammad Reza Mahmoudi <sup>a,b</sup>, Mohsen Maleki <sup>c</sup>, Kirill Borodin <sup>d,e</sup>,  
 Kim-Hung Pho <sup>f,\*</sup>, Dumitru Baleanu <sup>g,h</sup>

<sup>a</sup> Institute of Research and Development, Duy Tan University, Da Nang 550000, Vietnam

<sup>b</sup> Department of Statistics, Faculty of Science, Fasa University, Fasa, Fars, Iran

<sup>c</sup> Department of Statistics, University of Isfahan, Isfahan, Iran

<sup>d</sup> Department of Theoretical Physics and Applied Mathematics, Ural Federal University named after the first President of Russia B. N. Yeltsin, Yekaterinburg, Russian Federation

<sup>e</sup> M. N. Mikheev Institute of Metal Physics, Ural Branch, Russian Academy of Sciences, Yekaterinburg, Russian Federation

<sup>f</sup> Fractional Calculus, Optimization and Algebra Research Group, Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam

<sup>g</sup> Department of Mathematics, Faculty of Art and Sciences, Cankaya University Balgat 06530, Ankara, Turkey

<sup>h</sup> Institute of Space Sciences, Magurele-Bucharest, Romania

Received 4 December 2019; revised 13 March 2020; accepted 16 March 2020  
 Available online 14 May 2020

## KEYWORDS

Almost periodically correlated processes;  
 Almost cyclostationary processes;  
 Classification;  
 Clustering;  
 Spectral density

**Abstract** In time series analysis, comparing spectral densities of several processes with almost periodic spectra is an interested problem. The contribution of this work is to give a technique to compare and to cluster the spectral densities of some independent almost periodically correlated (cyclostationary) processes. This approach is based on the limiting distribution for the periodogram and the discrete Fourier transform. The real world examples and simulation results indicate that the approach well acts.

© 2020 The Authors. Published by Elsevier B.V. on behalf of Faculty of Engineering, Alexandria University. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

## 1. Introduction

Comparing spectral densities of several processes is an important topic that has many applications in signal processing, engineering, physics and other fields. The researchers like to explore if stochastic mechanism of some observed time series are similar or not. Many references have considered the clustering, classification and comparison of two or several time series. For example, the Refs. [1–25] studied these subjects for stationary time series. But the stationarity assumption is not satisfied in many situations, specially for the periodically

\* Corresponding author.

E-mail addresses: [mohammadrezamahmoudi@duytan.edu.vn](mailto:mohammadrezamahmoudi@duytan.edu.vn), [mahmoudi.m.r@fasau.ac.ir](mailto:mahmoudi.m.r@fasau.ac.ir) (M.R. Mahmoudi), [m.maleki.stat@gmail.com](mailto:m.maleki.stat@gmail.com) (M. Maleki), [borodin.kirilli@rambler.ru](mailto:borodin.kirilli@rambler.ru) (K. Borodin), [phokimhung@tdtu.edu.vn](mailto:phokimhung@tdtu.edu.vn) (K.-H. Pho), [dumitru@cankaya.edu.tr](mailto:dumitru@cankaya.edu.tr) (D. Baleanu).

Peer review under responsibility of Faculty of Engineering, Alexandria University.

<https://doi.org/10.1016/j.aej.2020.03.043>

1110-0168 © 2020 The Authors. Published by Elsevier B.V. on behalf of Faculty of Engineering, Alexandria University. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

rhythmic processes. Cyclostationary and almost cyclostationary (CS and ACS) time series can be naturally applied in these situations. The ACS time series are non-stationary processes with almost periodic mean and auto-correlation functions. These processes included CS and stationary processes. In view of frequency domain, the spectral square of ACS processes is supported with the main diagonal and parallel lines to the main diagonal,  $T_j(x) = x \pm \alpha_j, j = 1, 2, \dots$ , in spectral square  $[0, 2\pi) \times [0, 2\pi)$ . Different problems of ACS time series have been studied in previous researches [26–50]. Some statisticians considered the comparison of two CS time series [51–53]. Ref. [54] proposed an approach to detect the cycles in the movement in motion of a walking person using Fourier transform techniques and auto-correlation of the smoothed spatio-temporal curvature function. Ref. [55] reviewed and reported past researches that studied the time series data clustering in various fields. Ref. [56], using the log spectrum, introduced a new distance measure, called cepstral distance measure, to compare and cluster of some time series. Ref. [57] combined k-means clustering technique and discriminant analysis to introduce an iterative algorithm to classify the cyclic time series. Ref. [58] studied a new spectral clustering method, called transfer spectral clustering (TSC). Ref. [59] proposed an autoregressive tree-based ensemble approach, called multivariate autoregressive forest, to classify some multivariate processes.

In this research, we will apply the asymptotic distribution of the periodogram and discrete Fourier transform of ACS processes to construct an approach to compare and classify several ACS processes. Section 2 is devoted to notations and preliminaries. The technique to compare and classify the ACS processes is presented in Section 3. The ability of the introduced technique is investigated by means of extensive Monte Carlo simulations, and real world problems, in Sections 4 and 5, respectively.

## 2. Notations and preliminaries

A second order time series  $\{X_t : t \in \mathbb{Z}\}$  is an ACS time series if for every  $\tau \in \mathbb{Z}$ , its mean function,  $\mu(t) = E(X_t)$ , and its auto-covariance function,  $B(t, \tau) = \text{cov}(X_t, X_{t+\tau})$ , are almost periodic functions at  $t$  [50,60].

As Ref. [50], assume that the following assumptions are satisfied.

(A1)  $\{X_t : t \in \mathbb{Z}\}$  is a real-valued ACS time series with mean of zero.

By (A1), we have

$$B(t, \tau) = \sum_{\omega \in W_t} a(\omega, \tau) e^{i\omega t},$$

where

$$a(\omega, \tau) = \lim_{n \rightarrow \infty} \left( \frac{1}{n} \sum_{j=1}^n B(j, \tau) e^{-i\omega j} \right),$$

and for fixed  $\tau$ . Also as [60,61] indicated, the set  $W_\tau = \{\omega \in [0, 2\pi) : a(\omega, \tau) \neq 0\}$  is countable.

(A2) The set  $W = \bigcup_{\tau \in \mathbb{Z}} W_\tau$ , is finite and the spectral square of  $X_t$  is supported with the main diagonal and parallel lines

to the main diagonal,  $T_j(x) = x \pm \alpha_j, j = 1, 2, \dots$ , in spectral square  $[0, 2\pi) \times [0, 2\pi)$ .

Thus we have

$$B(t, \tau) = \sum_{\omega \in W} a(\omega, \tau) e^{i\omega t},$$

and the set  $S = \bigcup_{\omega \in W} \{(v, \gamma) \in [0, 2\pi)^2 : \gamma = v - \omega\}$  supports the spectral measure of  $X_t$ .

Moreover, the coefficients

$$a(\omega, \tau) = \int_0^{2\pi} e^{i\zeta\tau} r_\omega(d\zeta),$$

are the Fourier transforms of the measures  $r_\omega(\hat{A})$ .

(A3)  $r_0$  is an absolute continuous measure with respect to the Lebesgue measure.

The Ref. [61] indicated by considering this assumption and  $\sum_{\tau=-\infty}^{\infty} |a(\omega, \tau)| < \infty$ , for any  $\omega \in W$ , there exist a spectral density function  $f_\omega(\hat{A})$  such that

$$f_\omega(v) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} a(\omega, \tau) e^{-i v \tau}.$$

Therefore, we can represent an ACS time series, with finite number of cyclic frequencies, as

$$X_t = \int_0^{2\pi} e^{-itx} \zeta(dx), t \in \mathbb{Z},$$

where  $\zeta$  is a random spectral measure on  $[0, 2\pi)$  with the following property:

$$E\left(\zeta(d\theta) \zeta(d\theta')\right) = 0, (\theta, \theta') \notin S.$$

The spectral distribution matrix and the spectral density matrix of  $\zeta$ , are given by

$$F(d\lambda) = [F_{kj}(d\lambda)]_{j,k=1,\dots,m},$$

and

$$f(\lambda) = \frac{dF}{d\lambda} = [f_{kj}(\lambda)]_{j,k=1,\dots,m},$$

respectively, where

$$F_{kj}(d\lambda) = E\left(\zeta(d\lambda + \alpha_k) \zeta(d\lambda + \alpha_j)\right), k, j = 1, \dots, m,$$

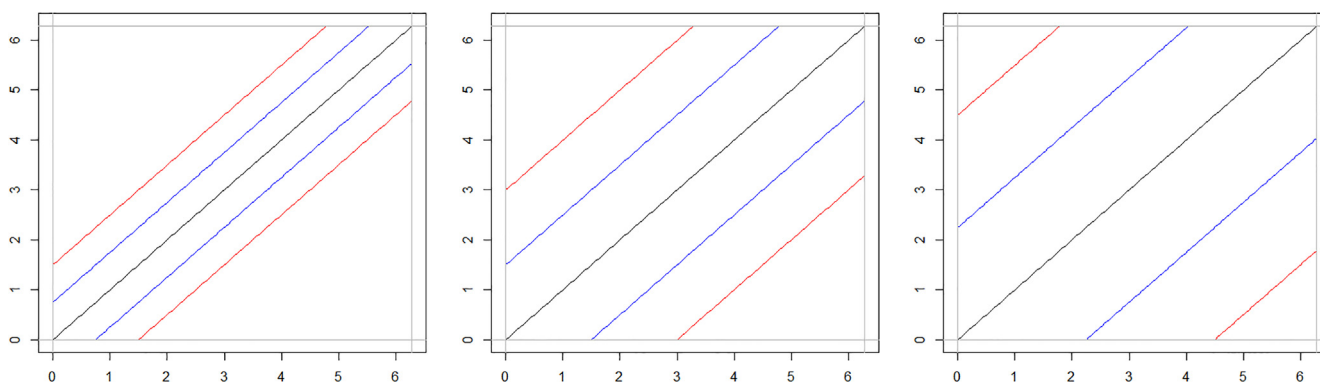
and  $f_{kj}$  is the spectral density function of  $F_{kj}$ .

Assume a sample  $X_0, \dots, X_{N-1}$ , from ACS process  $\{X_t : t \in \mathbb{Z}\}$ . The discrete Fourier transform (DFT) and periodogram of the observations  $X_0, \dots, X_{N-1}$ , are respectively given by

$$d_X(\lambda) = N^{-1/2} \sum_{t=0}^{N-1} X_t e^{-it\lambda}, \lambda \in [0, 2\pi),$$

and

$$I_X(\lambda) = |d_X(\lambda)|^2, \lambda \in [0, 2\pi).$$



**Fig. 1** The spectral square of the process, Left:  $\omega_i = 0.5$ , Middle:  $\omega_i = 1$ , and Right:  $\omega_i = 2$ .

The study the asymptotic properties of periodogram and DFT of ACS time series, see the Refs. [41–42,49,50].

### 3. Methodology

Suppose  $\{X_t^{(1)}, t = 1, \dots, n_1\}$ ,  $\{X_t^{(2)}, t = 1, \dots, n_2\}, \dots, \{X_t^{(l)}, t = 1, \dots, n_l\}$ , are  $l$  independent ACS processes with  $m$  spectral cycles.

Commonly, the researchers explore to test the null hypothesis  $H_0: X_t^{(1)} \equiv X_t^{(2)} \equiv \dots \equiv X_t^{(l)}$ , that is equivalent to  $H_0: f_1 = f_2 = \dots = f_l$ , such that  $f_1, \dots, f_l$  are the spectral

density matrices respectively corresponding to  $X_t^{(1)}, \dots$ , and  $X_t^{(l)}$ . If the null hypothesis  $H_0$  is not accepted then it can be concluded that at least two time series of the  $l$  time series have different rhythms, and if  $H_0$  is accepted consequently the stochastic behaviours of all time series are similar.

The periodogram for ACS time series is introduced [50] by

$$I_X^m(\lambda) = d_X^m(\lambda) d_X^{m*}(\lambda),$$

such that

$$d_X^m(\lambda) = (d_X(T_1(\lambda)), d_X(T_2(\lambda)) \dots, d_X(T_m(\lambda)))', \lambda \in [0, 2\pi),$$

**Table 1** The values of  $\hat{\alpha}$  and  $\hat{\pi}$  for the introduced approach.

			$(n_1, n_2, n_3)$			
$\omega_1$	$\omega_2$	$\omega_3$	(100, 50, 75)	(150, 75, 100)	(200, 150, 100)	(500, 250, 300)
0.75	0.75	0.75	0.052	0.050	0.050	0.049
		1.5	0.736	0.845	0.909	0.976
		2.25	0.727	0.828	0.925	0.978
	1.5	0.75	0.706	0.837	0.907	0.999
		1.5	0.760	0.864	0.959	0.994
		2.25	0.766	0.866	0.955	0.987
	2.25	0.75	0.718	0.812	0.903	0.970
		1.5	0.758	0.889	0.969	0.985
		2.25	0.800	0.883	0.945	0.986
1.5	0.75	0.75	0.772	0.864	0.969	1.000
		1.5	0.727	0.842	0.939	0.972
		2.25	0.760	0.861	0.946	0.989
	1.5	0.75	0.741	0.842	0.917	0.996
		1.5	0.051	0.051	0.050	0.050
		2.25	0.725	0.845	0.906	0.970
	2.25	0.75	0.782	0.897	0.963	0.983
		1.5	0.717	0.835	0.919	0.993
		2.25	0.790	0.880	0.943	0.997
2.25	0.75	0.75	0.791	0.888	0.952	0.984
		1.5	0.777	0.857	0.963	0.999
		2	0.729	0.847	0.904	0.994
	1.5	0.75	0.779	0.870	0.965	0.989
		1.5	0.797	0.871	0.959	0.989
		2.25	0.718	0.835	0.933	0.978
	2.25	0.75	0.718	0.844	0.904	0.976
		1.5	0.724	0.844	0.933	0.973
		2.25	0.052	0.051	0.049	0.049

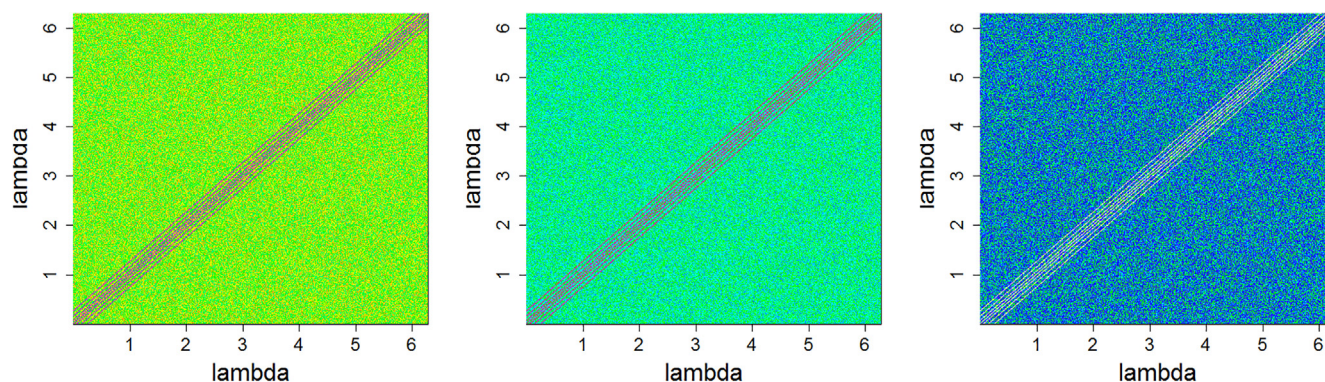


Fig. 2 Spectral frequency square (Left: Part 1, Middle: Part 2, Right: Part 3).

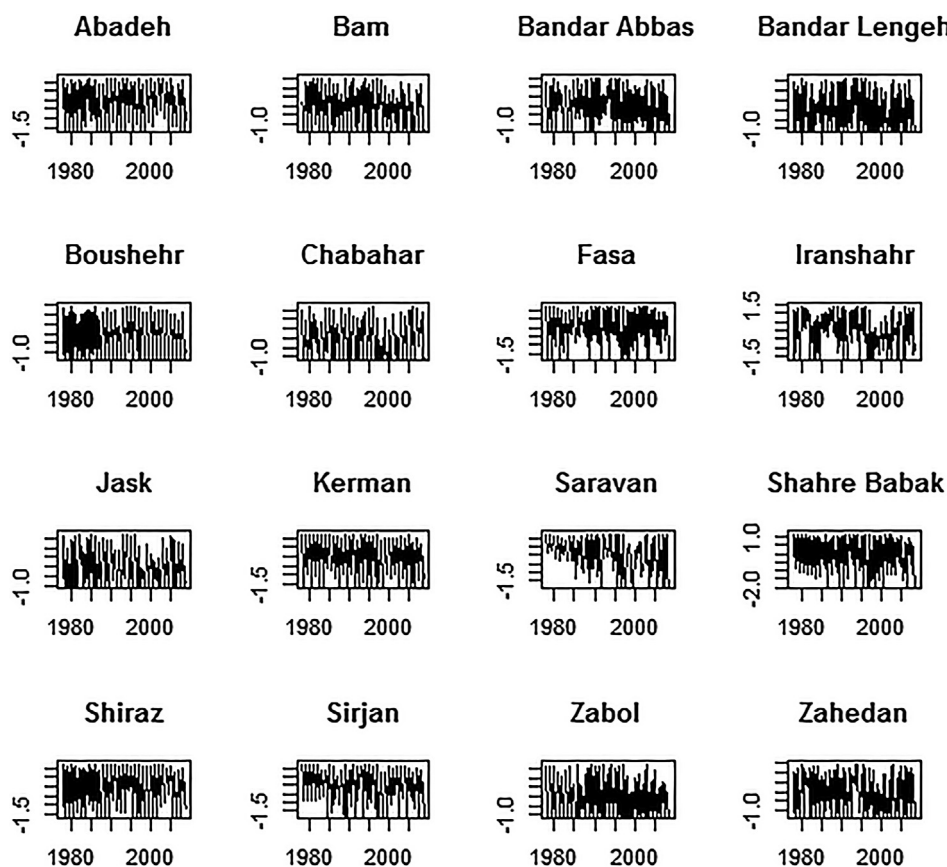


Fig. 3 Seasonal RDI time series of 16 Iranian synoptic stations (1980–2010).

**Table 2** Testing the equality of different parts.

Test Statistic	P-Value
$\chi^2_{*} = 4.193$	0.651

where  $\mathbf{d}_X^{m*}(\lambda)$  is the complex conjugate transpose of  $\mathbf{d}_X^m(\lambda)$ .

**Lemma 3.1.** Let  $\{X_t, t \in \mathbb{Z}\}$  is an ACS time series with corresponding spectral density  $f(\lambda), \lambda \in [0, 2\pi)$ . Assume the frequencies  $\lambda_1 < \dots < \lambda_J \in [0, 2\pi)$ . Then

- (i)  $f(\lambda)$  can be asymptotically estimated by  $\hat{f}(\lambda) := \frac{F_X^m(\lambda)}{2\pi}$ .
- (ii)  $\mathbf{d}_X^m(\lambda_j), j = 1, \dots, J$ , have the asymptotic and independent  $m$ -variate complex normal distributions,  $N_m^c(0, 2\pi f(\lambda_j))$ .
- (iii)  $\mathbf{F}_X^m(\lambda_j), j = 1, \dots, J$ , have the asymptotic and independent complex Wishart distributions,  $W_m^c(1, 2\pi f(\lambda_j))$ .

**Proof:** ([50]).  $\square$

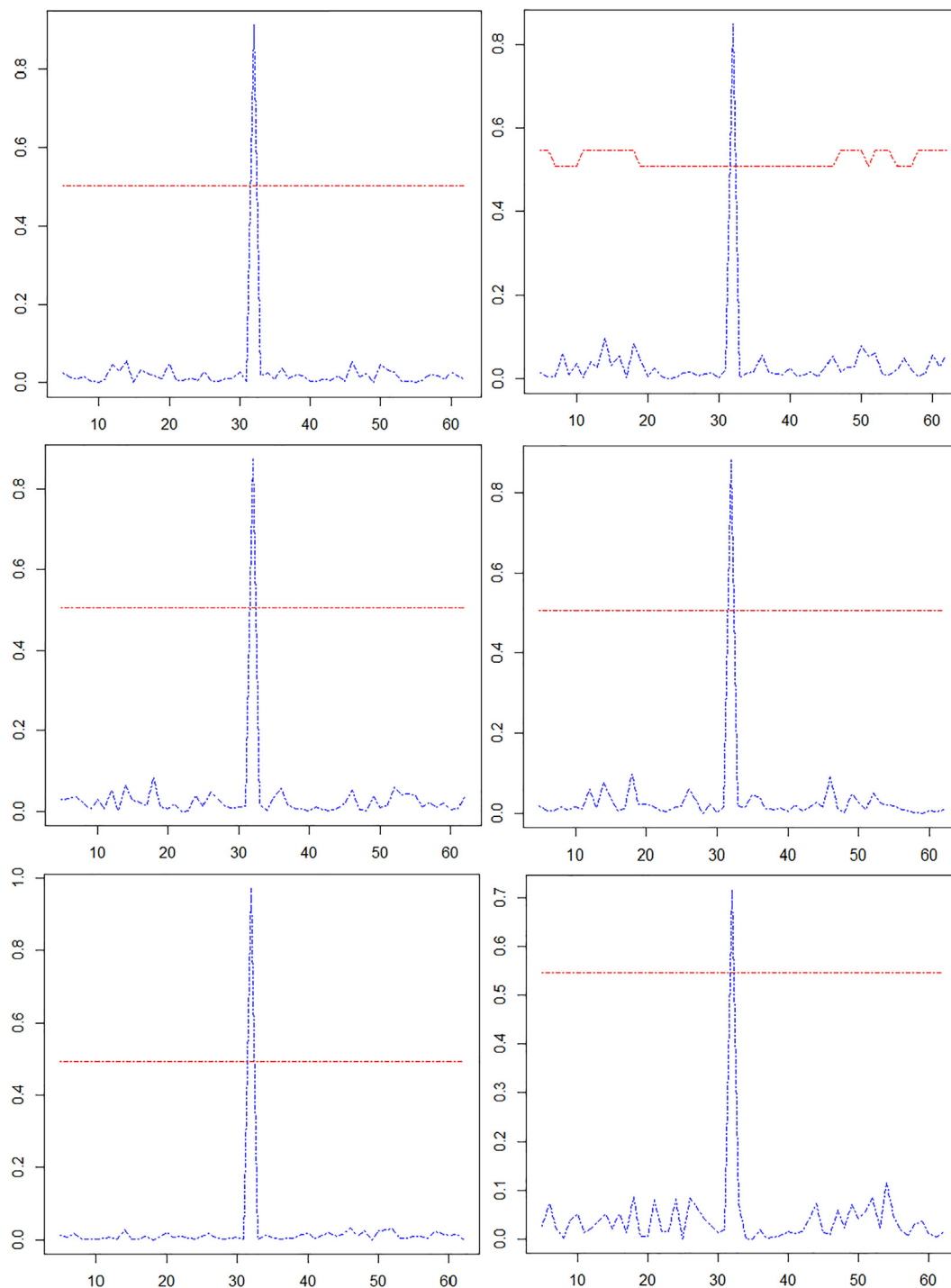
Let  $Y_j^{(k)} = \text{Re}(\mathbf{d}_{X^{(k)}}^m(\lambda_j)), j = 1, \dots, J, k = 1, 2, \dots, l$ , and  $Z_j^{(k)} = \text{Im}(\mathbf{d}_{X^{(k)}}^m(\lambda_j)), j = 1, \dots, J, k = 1, 2, \dots, l$ , such that  $\mathbf{d}_{X^{(k)}}^m(\lambda_j)$  is the  $\mathbf{d}_X^m(\lambda_j)$  corresponding to  $k^{\text{th}}$  time series. As a

result of Lemma 3.1, it can be concluded that for  $k = 1, 2, \dots, l$ , the asymptotic distribution of  $W_j^{(k)} = \left( Y_j^{(k)}, Z_j^{(k)} \right)'$  is  $N_{2m}(0, \Sigma_j^{(k)})$ , such that  $\Sigma_j^{(k)} = \begin{bmatrix} V_{Y_j Y_j}^{(k)} & V_{Y_j Z_j}^{(k)} \\ V_{Z_j Y_j}^{(k)} & V_{Z_j Z_j}^{(k)} \end{bmatrix}$ ,  $V_{AB} = \text{COV}(A, B)$ . Consequently, the asymptotic distribution of  $U^{(k)} = \sum_{j=1}^J W_j^{(k)}$  is  $N_{2m}(0, \Sigma^{(k)})$ , such that

$$\Sigma^{(k)} = \Sigma_1^{(k)} + \dots + \Sigma_J^{(k)}.$$

### 3.1. Test of hypothesis

As previous discussion, usually, the researchers want to test the null hypothesis  $H_0 : f_1 = f_2 = \dots = f_l$ , that is equivalent to  $H_0 : \Sigma^{(1)} = \Sigma^{(2)} = \dots = \Sigma^{(l)}$ . It can be concluded that the



**Fig. 4** Periodogram for seasonal RDI time series of Iranian synoptic stations (1980–2010). Left: Top: Abadeh, Middle: Bandar Abbas, Bottom: Boushehr. Right: Top: Bam, Middle: Bandar Lengeh, Bottom: Chabahar. Left: Top: Fasa, Middle: Jask, Bottom: Saravan. Right: Top: Iranshahr, Middle: Kerman, Bottom: Shahre Babak. Left: Top: Shiraz, Bottom: Zabol. Right: Top: Sirjan, Bottom: Zahedan.



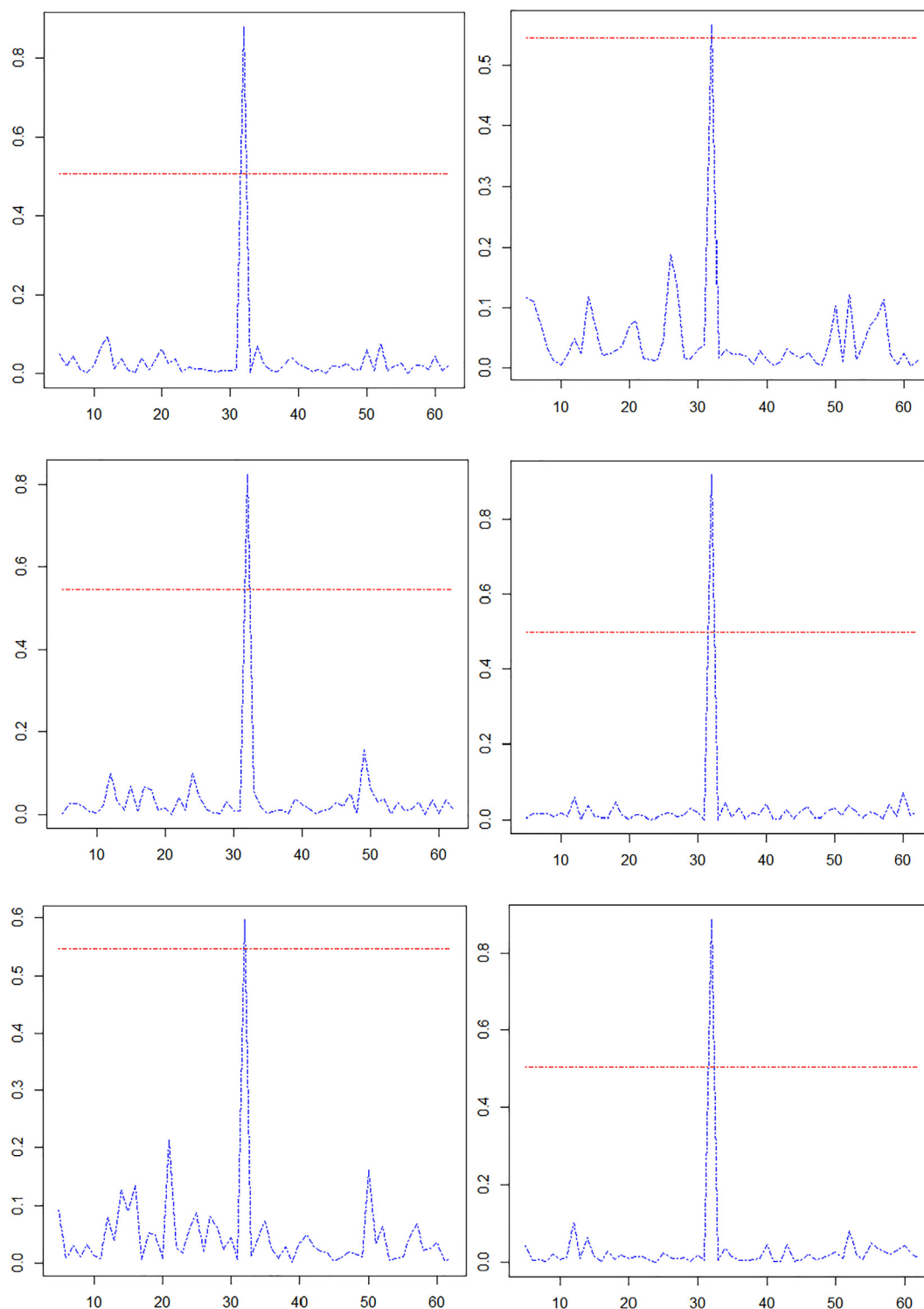


Fig. 4 (continued)

asymptotic distribution of  $U = U^{(1)} + U^{(2)} + \dots + U^{(l)}$  is  $N_{2m}(0, \Sigma)$ , such that  $\Sigma = \Sigma^{(1)} + \Sigma^{(2)} + \dots + \Sigma^{(l)}$ .

Therefore the asymptotic distribution of statistic

$$\chi^2 = (U)'(\Sigma)^{-1}(U),$$

is chi-square with  $2m$  degrees of freedom,  $\chi^2(2m)$ .

The asymptotic distribution of statistic  $\chi^2$  can be applied to establish test of hypothesis about the null hypothesis  $H_0$ . As can be seen, the statistic  $\chi^2$  is related to the unknown parameter  $\Sigma$ .

**Remark 3.1.** To produce a sample of size  $N_i$  for the discrete Fourier transform of  $i^{th}$  process, different bootstrap approaches can be applied. In this research we used the

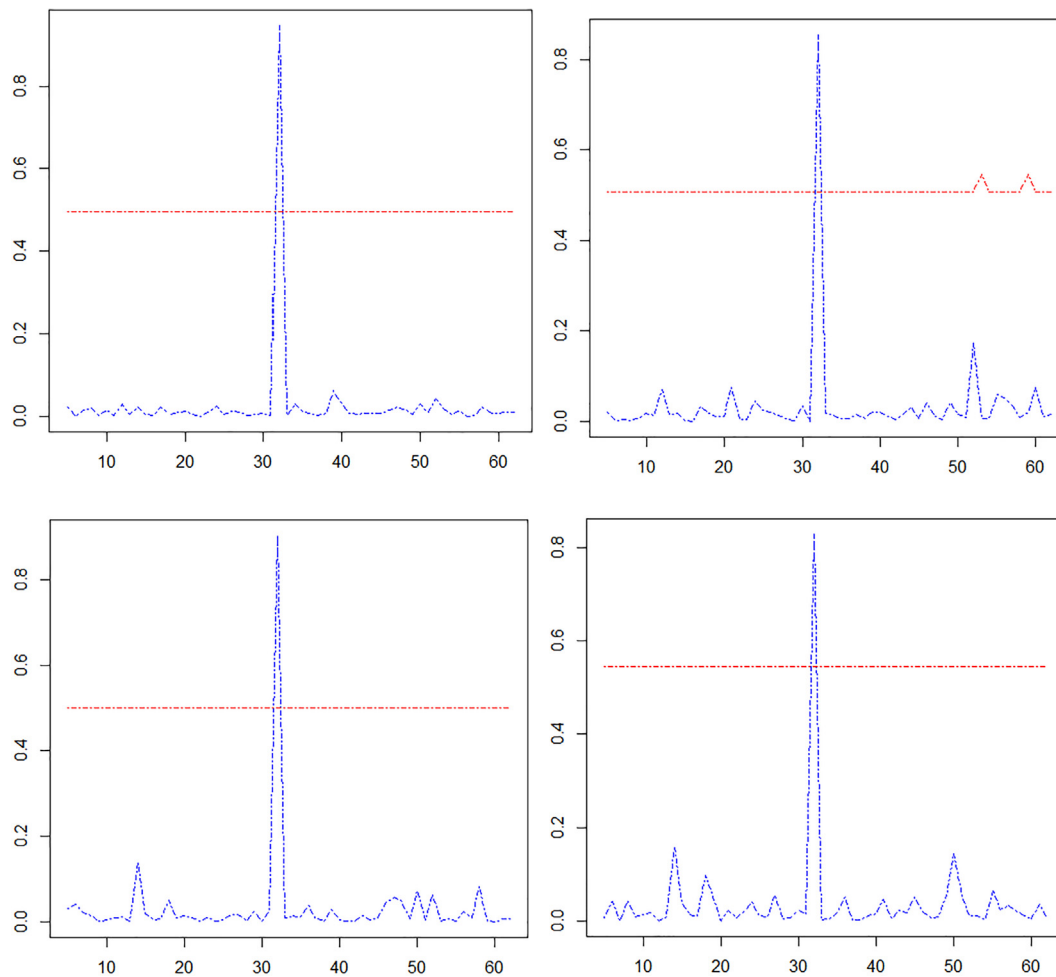


Fig. 4 (continued)

moving block bootstrap (MBB) (more details are given in [62]).

Let  $\mathbf{S} = \frac{(N_1-1)\mathbf{S}^{(1)} + (N_2-1)\mathbf{S}^{(2)} + \dots + (N_l-1)\mathbf{S}^{(l)}}{N_1 + N_2 + \dots + N_l - l}$ , as the pooled covariance matrix of  $\mathbf{S}^{(1)}, \dots, \mathbf{S}^{(l)}$ , such that  $\mathbf{S}^{(k)} = \mathbf{S}_1^{(k)} + \dots + \mathbf{S}_J^{(k)}$ ,  $\mathbf{S}_j^{(k)} = \begin{bmatrix} \hat{\mathbf{V}}_{Y_j Y_j}^{(k)} & \hat{\mathbf{V}}_{Y_j Z_j}^{(k)} \\ \hat{\mathbf{V}}_{Z_j Y_j}^{(k)} & \hat{\mathbf{V}}_{Z_j Z_j}^{(k)} \end{bmatrix}$ , and  $\hat{\mathbf{V}}_{AB} = \widehat{COV}(A, B)$ . If the null hypothesis  $H_0 : \Sigma^{(1)} = \Sigma^{(2)} = \dots = \Sigma^{(l)}$  be true, then the covariance matrix  $\Sigma$  can be consistently estimated by  $\mathbf{S}$ , and consequently by using the Weak Law of Large Numbers, the asymptotic distribution of the test statistic

$$\chi^{2*} = (\mathbf{U})' (\mathbf{S})^{-1} (\mathbf{U}),$$

is  $\chi^2(2m)$ . Therefore, for a given size  $\alpha$ ,  $H_0$  is rejected if  $\chi^{2*} > \chi_\alpha^2(2m)$ .

### 3.2. Clustering and classifying the processes

In Section 3.1, the rejection of  $H_0$  is equivalent to that at least two time series significantly differ from each other. Multiple comparisons should be applied to compare each pair of observed ACS time series.

Therefore, to classify and cluster the ACS processes, the following can be applied:

*Step 1:* First, we test the null hypothesis  $H_0 : \mathbf{f}_1 = \mathbf{f}_2 = \dots = \mathbf{f}_l$ .

If  $H_0$  is not rejected, consequently the stochastic behaviours of all processes are similar and we have only one cluster contained all of time series  $X_t^{(1)}, \dots$ , and  $X_t^{(l)}$ , and the clustering procedure is stopped.

If the null hypothesis  $H_0$  is rejected then it can be concluded that at least two time series of the  $l$  time series have different rhythms, and we will continue next steps.

*Step 2:* For  $i, j = 1, \dots, l$ , we test the null hypothesis  $H_0 : \mathbf{f}_i = \mathbf{f}_j$  and compute the P-values.

*Step 3:* If the comparison test of the  $i^{th}$  and  $j^{th}$  processes is not significant (P-value greater than 0.05), then there is no significant difference between two processes and they fall into one cluster. If else (P-value < 0.05), these two processes significantly differ from each other and consequently they fall into separate clusters.

*Step 4:* The APC time series are clustered based on all comparisons.

#### 4. Simulation study

To analyze the accuracy of proposed method, we generated  $(n_1, n_2, n_3) = \{(100, 50, 75), (150, 75, 100), (200, 150, 100), (500, 250, 300)\}$

observations from the ACS processes

$$X_t^{(i)} = (1 + \cos(\omega_i t)) Y_t^{(i)}, \omega \in (0, \infty), \quad i = 1, 2, 3,$$

where

$$Y_t^{(i)} = Z_t^{(i)} + 0.5Z_{t-1}^{(i)},$$

and  $Z_t^{(i)}, i = 1, 2, 3$ , are independent sequences of IIDN(0,1).

The spectral supports of  $X_t^{(i)}$  is as following lines:

$$\begin{aligned} T_1(x) &= x, T_2(x) = x + \omega_i, T_3(x) = x - \omega_i, T_4(x) \\ &= x - 2\omega_i, T_5(x) = x + 2\omega_i. \end{aligned}$$

Fig. 1 indicates the spectral plane  $[0, 2\pi)^2$ , for  $\omega_i = \{0.75, 1.5, 2.25\}$ .

First, we estimated the Type I error probability ( $\hat{\alpha}$ ) and power ( $\hat{\pi}$ ) based on 1000 replications and 1000 iterations, as the percent of iterations that the assumption  $H_0$  is rejected. Then we graph Q-Q plot for the test statistic  $\chi^{2*}$  based on the computed values of the simulation runs.

Table 1 report the values of  $\hat{\alpha}$  (in rows: 1th, 14th and 27th) and  $\hat{\pi}$  (other rows). The results indicates that the values of  $\hat{\alpha}$  is very close to the considered size ( $\alpha = 0.05$ ), especially when  $(n_1, n_2, n_3)$  grows. Also the power studies show that the proposed method excellently discriminate  $H_0$  from  $H_1$ .

#### 5. Real data

This section is devoted to illustrate the ability of introduced in practical cases.

**Example 1.** The dataset includes the first difference of centered moving average filter  $2 \times 12$  moving average (MA) applied for logarithm of industrial production index (IPI) in Poland (2005 = 100%) since January 1995 untile December 2009, [43].

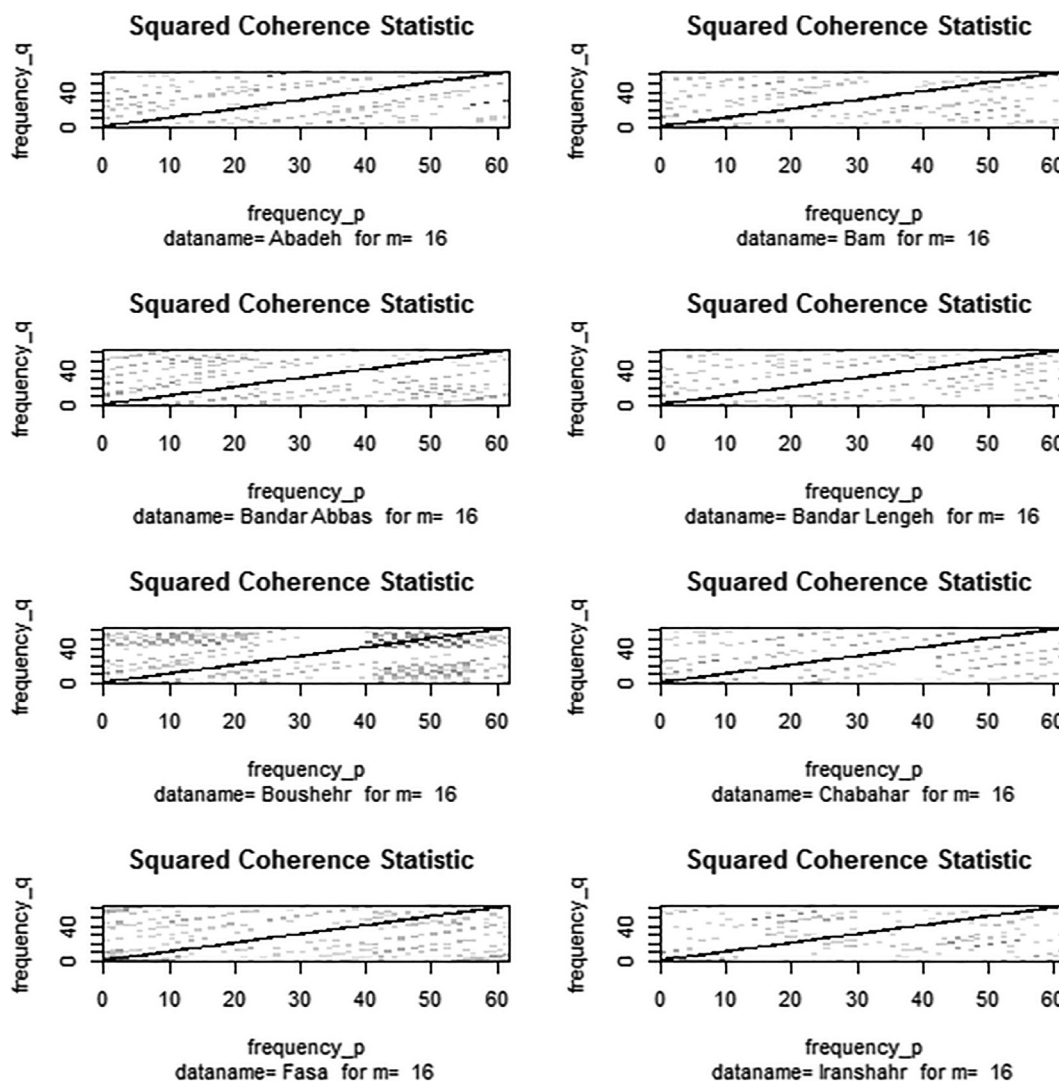


Fig. 5 Spectral frequency square for seasonal RDI time series of 16 Iranian synoptic stations (1980–2010).



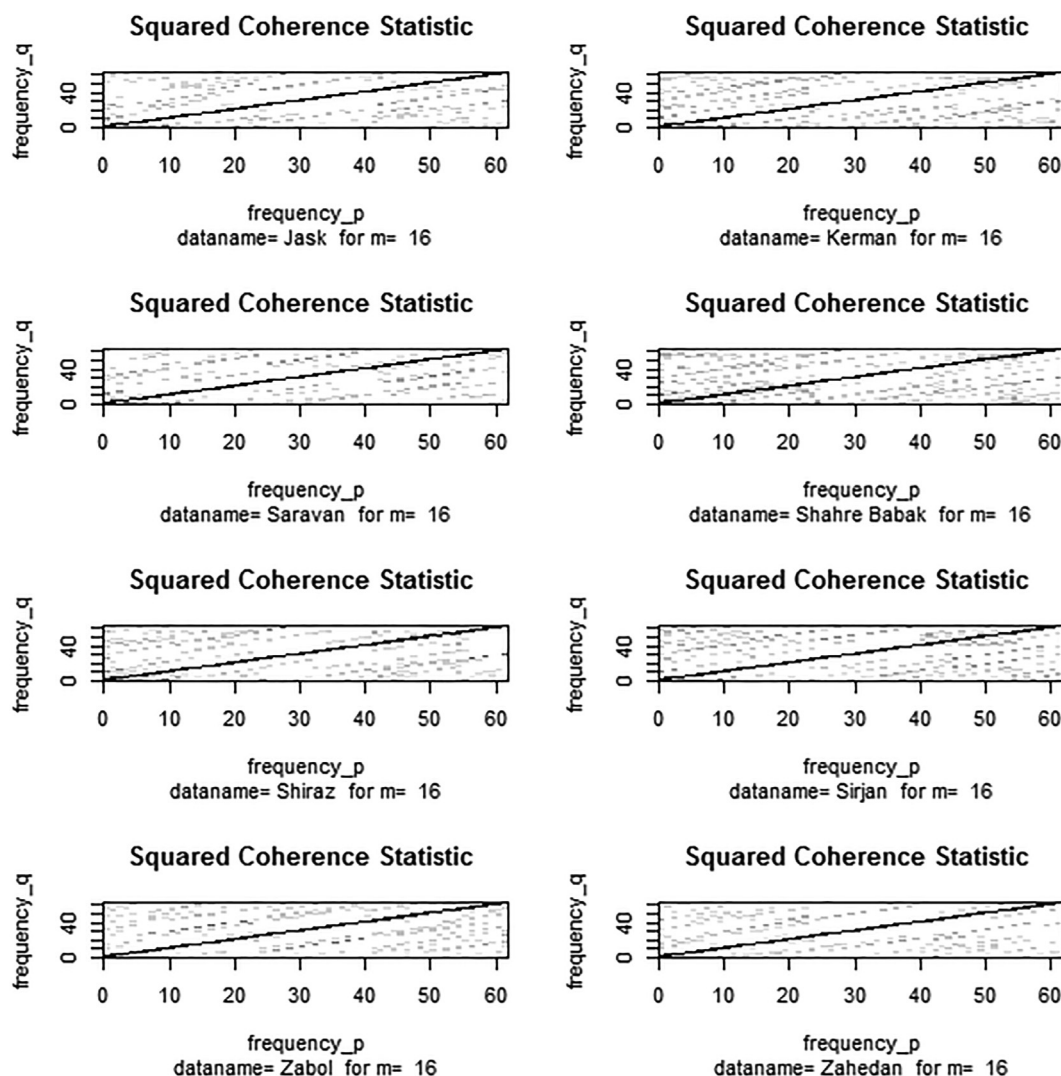


Fig. 5 (continued)

**Table 3** Testing the equality of seasonal RDI time series of 16 Iranian synoptic stations (1980–2010).

Test Statistic	P-Value
$\chi^2_{*} = 22.545$	0.004

We split this dataset in three parts with equal sizes. The spectral frequency squares of these parts are given in Fig. 2. The results detect ACS time series with spectra on the lines  $T_j(x) = x \pm \alpha$ ,  $\alpha \in \{0.062, 0.153, 0.258\}$ . This result verifies the given result in [43]. Then the introduced technique is used to test the hypothesis  $\Sigma^{(1)} = \Sigma^{(2)} = \Sigma^{(3)}$  (or equivalently,  $f_1 = f_2 = f_3$ ). Table 2 summarizes the results. As can be seen, since the P-value is more than 0.05, thus the null hypothesis cannot be rejected and consequently the stochastic behaviours of all processes are similar.

**Example 2.** In agricultural and environmental sciences, the drought is an important subject. Droughts have many effects on crop, urban water supply, degradation and desertification.

**Table 4** Multiple comparisons between seasonal RDI time series of 16 Iranian synoptic stations (1980–2010).

Cluster 1	Cluster 2	Cluster 3
Abadeh	Bam	Chabahar
Boushehr	Bandar Abbas	Iranshahr
Fasa	Bandar Lengeh	Saravan
Kerman	Jask	
Shahre Babak	Zabol	
Shiraz	Zahedan	
Sirjan		

To evaluate the rate of drought, researchers introduced many different drought indices [63–65]. In this work, the datasets contain the seasonal reconnaissance drought index (RDI) values from 16 Iranian synoptic stations from 1980 to 2010 (Fig. 3). The periodogram and the spectral frequency squares of these datasets are given in Figs. 4 and 5, respectively. As can be seen in Fig. 4, the periodogram of all datasets detect PC processes with periods 4 ( $T = 124/31 = 4$ ,  $p < 0.05$ ). This

result also confirms by the spectral frequency squares of these datasets, where the spectral squares are supported by the lines  $T_j(x) = x \pm \frac{2\pi j}{4}, j = 0, \dots, 3$ .

Now, the introduced technique is used to test the hypothesis  $\Sigma^{(1)} = \dots = \Sigma^{(16)}$  (or equivalently,  $f_1 = \dots = f_{16}$ ). Table 3 summarizes the results. As can be seen, since the P-value is  $< 0.05$ , thus the null hypothesis is rejected and consequently at least the stochastic behaviours of two processes are different.

Now, we are interested in clustering these stations. To response this question, the proposed approach has been applied. Table 4 summarized the result for the multiple comparisons. As Table 4 indicates, these stations can be classified in 2 clusters; Cluster 1: Abadeh, Boushehr, Fasa, Kerman, Shahre Babak, Shiraz, Sirjan; Cluster 2: Bam, Bandar Abbas, Bandar Lengeh, Jask, Zabol, Zahedan; Cluster 3: Chabahar, Iranshahr, Saravan.

### Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

### References

- [1] P. De Souza, P. Thomson, Lpc distance measures and statistical tests with particular reference to likelihood ratio, *IEEE Trans. Acoust. Speech Signal Process* 30 (1982) 304–315.
- [2] D.S. Coates, P.J. Diggle, Tests for comparing two estimated spectral densities, *J. Time Ser. Anal.* 7 (1986) 7–20.
- [3] B. Potscher, E. Reschenhofer, Discriminating between two spectral densities in case of replicated observations, *J. Time Ser. Anal.* 9 (1988) 221–224.
- [4] P.J. Diggle, N.I. Fisher, Nonparametric Comparison of Cumulative Periodograms, *Appl. Stat.* 40 (1991) 423–434.
- [5] G.R. Dargahi-Noubary, Discrimination between Gaussian time series based on their spectral differences, *Commun. Stat. (Theory and Methods)* 21 (1992) 2439–2458.
- [6] P.J. Diggle, I. Wasel, Spectral analysis of replicated biomedical time series, *Appl. Statistics* 46 (1997) 31–71.
- [7] Y. Kakizawa, R.H. Shumway, M. Taniguchi, Discrimination and Clustering for Multivariate Time series, *J. Am. Stat. Assoc.* 93 (1998) 328–340.
- [8] J. Timmer, M. Lauk, W. Vach, C.H. Lucking, A test for the difference between spectral peak frequencies, *Comput. Stat. Data Anal.* 30 (1999) 45–55.
- [9] E.A. Maharaj, Comparison and classification of stationary multivariate time series, *Pattern Recogn.* 32 (1999) 1129–1138.
- [10] E.A. Maharaj, Clusters of time series, *J. Classif.* 17 (2000) 297–314.
- [11] E.A. Maharaj, Comparison of non-stationary time series in the frequency domain, *Comput. Stat. Data Anal.* 40 (2002) 131–141.
- [12] E.A. Maharaj, Using wavelets to compare time series patterns, *Int. J. Wavelets Multiresolut. Inf. Process.* 3 (2005) 511–521.
- [13] J. Caiado, N. Crato, D. Pena, A Periodogram-based Metric for Time Series Classification, *Comput. Stat. Data Anal.* 50 (2006) 2668–2684.
- [14] M. Eichler, Testing nonparametric and semiparametric hypotheses in vector stationary processes, *J. Multivariate Anal.* 99 (2008) 968–1009.
- [15] K. Fokianos, A. Savvides, On comparing several spectral densities, *Technometrics* 50 (3) (2008) 317–331.
- [16] J. Caiado, N. Crato, D. Pena, Comparison of times series with unequal length in the frequency domain, *Comm. Statist. Simulation Comput.* 38 (2009) 527–540.
- [17] H. Dette, E. Paparoditis, Bootstrapping frequency domain tests in multivariate time series with an application to comparing spectral densities, *J. R. Statistical Soc. B* 71 (2009) 831–8571.
- [18] H. Dette, T. Kinsvater, M. Vetter, Testing nonparametric hypotheses for stationary processes by estimating minimal distances, *J. Time Ser. Anal.* 32 (2010) 447–461.
- [19] H. Dette, T. Hildebrandt, A note on testing hypothesis for stationary processes in the frequency domain, *J. Multivariate Anal.* 104 (2011) 101–114.
- [20] C. Jentsch, A new frequency domain approach of testing for covariance stationarity and for periodic stationarity in multivariate linear processes, *J. Time Series Anal.* 33 (2012) 177–192.
- [21] C. Jentsch, M. Pauly, A note on using periodogram-based distances for comparing spectral densities, *Statist. Probab. Lett.* 82 (2012) 158–164.
- [22] G.E. Salcedo, R.F. Porto, P.A. Morettin, Comparing non-stationary and irregularly spaced time series, *Comput. Stat. Data Anal.* 56 (12) (2012) 3921–3934.
- [23] C. Jentsch, M. Pauly, Testing equality of spectral densities using randomization techniques, *Bernoulli* 21 (2) (2015) 697–739.
- [24] U. Triacca, Measuring the distance between sets of ARMA models, *Econometrics* 4 (3) (2016) 32.
- [25] M.R. Mahmoudi, M. Maleki, A. Pak, Testing the difference between two independent time series models, *Iranian J. Sci. Technol.: Sci.* 41 (3) (2017) 665–669.
- [26] E.G. Gladyshev, Periodically correlated random sequences, *Soviet Math. Dokl.* 2 (1961) 385–388.
- [27] E.G. Gladyshev, Periodically and almost periodically correlated random processes with a continuous time parameter, *Theory Probab. Appl.* 8 (1963) 173–177.
- [28] W.A. Gardner, Exploitation of spectral redundancy in cyclostationary signals, *IEEE Signal Process Mag.* 8 (2) (1991) 14–36.
- [29] H. Hurd, Correlation theory of almost periodically correlated processes, *J. Multivariate Anal.* 37 (1991) 24–45.
- [30] H. Hurd, J. Leskow, Strongly consistent and asymptotically normal estimation of the covariance for almost periodically correlated processes, *Statist. Decisions* 10 (1992) 201–225.
- [31] J. Leskow, A. Weron, Ergodic behavior and estimation for periodically correlated processes, *Statist. Probab. Lett.* 15 (1992) 299–304.
- [32] W.A. Gardner (Ed.), *Cyclostationarity in Communications and Signal Processing*, IEEE Press, New York, 1994.
- [33] J. Leskow, Asymptotic normality of the spectral density estimator for almost periodically correlated stochastic processes, *Stoch. Process. Appl.* 52 (1994) 351–360.
- [34] K.-S. Lii, M. Rosenblatt, Spectral analysis for harmonizable processes, *Ann. Statist.* 30 (1) (2002) 258–297.
- [35] K.-S. Lii, M. Rosenblatt, Estimation for almost periodic processes, *Ann. Statist.* 34 (3) (2006) 1115–1139.
- [36] W.A. Gardner, A. Napolitano, L. Paura, Cyclostationarity: half a century of research, *Signal Process.* 86 (2006) 639–697.
- [37] H.L. Hurd, A.G. Miamee, *Periodically Correlated Random Sequences: Spectral Theory and Practice*, Wiley, Hoboken, 2007.
- [38] L. Lenart, Asymptotic properties of periodogram for almost periodically correlated time series, *Prob. Math. Stat.* 28 (2) (2008) 305–324.
- [39] L. Lenart, Asymptotic distributions and subsampling in spectral analysis for almost periodically correlated time series, *Bernoulli* 17 (1) (2011) 290–319.
- [40] A. Napolitano, *Generalizations of Cyclostationary Signal Processing: Spectral Analysis and Applications*, Wiley-IEEE Press, 2012.

- [41] L. Lenart, Non-parametric frequency identification and estimation in mean for almost periodically correlated time series, *J. Multivariate Anal.* 115 (2013) 252–269.
- [42] L. Lenart, M. Pipien, Seasonality revisited - statistical testing for almost periodically correlated processes, *Central Eur. J. Econ. Model. Economet.* 5 (2013) 85–102.
- [43] L. Lenart, M. Pipien, Almost periodically correlated time series in business fluctuations analysis, *Acta Phys. Pol. A* 123 (3) (2013) 567–583.
- [44] M.R. Mahmoudi, A.R. Nematollahi, A.R. Soltani, On the detection and estimation of simple processes, *Iranian J. Sci. Technol., A* 39 (A2) (2015) 239–242.
- [45] A. Napolitano, Cyclostationarity: limits and generalizations, *Signal Process.* 120 (2016) 323–347.
- [46] A. Napolitano, Cyclostationarity: new trends and applications, *Signal Process.* 120 (2016) 385–408.
- [47] M.R. Mahmoudi, M. Maleki, A new method to detect periodically correlated structure, *Comput. Statistics* 32 (4) (2017) 1569–1581.
- [48] A.R. Nematollahi, A.R. Soltani, M.R. Mahmoudi, Periodically correlated modeling by means of the periodograms asymptotic distributions, *Stat. Pap.* 58 (4) (2017) 1267–1278.
- [49] L. Lenart, M. Pipien, Non-parametric test for the existence of the common deterministic cycle: the case of the selected European countries, *Central Eur. J. Econ. Modeling Economet.* 9 (3) (2017) 201–241.
- [50] M.R. Mahmoudi, M.H. Heydari, Z. Avazzadeh, On the asymptotic distribution for the periodograms of almost periodically correlated (cyclostationary) processes, *Digital Signal Process.* 81 (2018) 186–197.
- [51] M.R. Mahmoudi, M.H. Heydari, Z. Avazzadeh, Testing the difference between spectral densities of two independent periodically correlated (cyclostationary) time series models, *Commun. Statistics—Theory Methods* 48 (9) (2018) 2320–2328.
- [52] M.R. Mahmoudi, M.H. Heydari, R. Roohi, A new method to compare the spectral densities of two independent periodically correlated time series, *Math. Comput. Simul.* 160 (2018) 103–110.
- [53] M.R. Mahmoudi, M.H. Heydari, Z. Avazzadeh, K.H. Pho, Goodness of fit test for almost cyclostationary processes, *Digital Signal Process.* 96 (2020) 102597.
- [54] P.S. Tsai, M. Shah, K. Keiter, T. Kasparis, Cyclic motion detection for motion based recognition, *Pattern Recogn.* 27 (12) (1994) 1591–1603.
- [55] T.W. Liao, Clustering of time series data—a survey, *Pattern Recogn.* 38 (11) (2005) 1857–1874.
- [56] A. Savvides, V.J. Promponas, K. Fokianos, Clustering of biological time series by cepstral coefficients based distances, *Pattern Recogn.* 41 (7) (2008) 2398–2412.
- [57] A. Gharehbaghi, P. Ask, A. Babic, A pattern recognition framework for detecting dynamic changes on cyclic time series, *Pattern Recogn.* 48 (3) (2015) 696–708.
- [58] W. Jiang, W. Liu, F.L. Chung, Knowledge transfer for spectral clustering, *Pattern Recogn.* 81 (2018) 484–496.
- [59] K.S. Tuncel, M.G. Baydogan, Autoregressive forests for multivariate time series modeling, *Pattern Recogn.* 73 (2018) 202–215.
- [60] C. Corduneanu, *Almost Periodic Functions*, Chelsea, New York, 1989.
- [61] D. Dehay, H. Hurd, Representation and estimation for periodically and almost periodically correlated random processes, in: W.A. Gardner (Ed.), *Cyclostationarity in Communications and Signal Processing*, IEEE Press, 1994, pp. 295–329.
- [62] R. Synowiecki, Consistency and application of moving block bootstrap for nonstationary time series with periodic and almost periodic structure, *Bernoulli* 13 (2007) 1151–1178.
- [63] A.R. Zarei, M.R. Mahmoudi, Evaluation of changes in RDIST index effected by different Potential Evapotranspiration calculation methods, *Water Resour. Manage.* 31 (15) (2017) 4981–4999.
- [64] A.R. Zarei, M.M. Moghimi, M.R. Mahmoudi, Analysis of changes in spatial pattern of drought using RDI index in south of Iran, *Water Resour. Manage.* 30 (11) (2016) 3723–3743.
- [65] A.R. Zarei, M.M. Moghimi, M.R. Mahmoudi, Parametric and Non-parametric trend of drought in arid and semi-arid regions using RDI index, *Water Resour. Manage.* 30 (14) (2016) 5479–5500.